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GENERAL POSITION MAPS FOR TOPOLOGICAL MANIFOLDS IN THE 2/3RDS RANGE(1)

BY

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ABSTRACT. For each proper map f of a topological m-manifold M into a topological q-manifold Q, $m \leqslant (2/3)q - 1/3$, we build an approximating map g such that the set of singularities S of g is a locally finite simplicial $(2m - q)$-complex locally tamely embedded in M, $g(S)$ is another locally finite complex $g|: S \twoheadrightarrow g(S)$ is a piecewise linear map and g is a locally flat embedding on the complement of S.

Furthermore if $f|\partial M$ is a locally flat embedding then we construct g so that it agrees with f on ∂M even when $f(\partial M)$ meets Int $Q \cap f(\text{Int } M)$.

In addition we present two other general position lemmas. Also, we show that given two codimension $\geqslant 3$ locally flat topological submanifolds M and V of a topological manifold Q, dim M + dim V − dim $Q \leqslant 3$, then we can move M so that M and V are transverse in Q.

1. Introduction. In this paper we shall define "general position" for the topological category and we shall establish some general position lemmas. The only tool that we shall use for this is the codim $\geqslant 3$ Taming Lemma 2.2 of Bryant, Seebeck, Černavskiĭ, Homma and Miller.

We list the results and definitions:

DEFINITION. Let $g: M \to Q$ be a continuous map of an m-manifold into a q-manifold. We say that g is in *general position* if there exists locally finite complexes K_m and K_q, which are closed subsets of M and Q, respectively, such that

(i) K_m is the singular set of g,

(ii) $\dim K_m \leqslant 2m - q$,

(iii) g sends K_m piecewise linearly onto K_q, and

(iv) $g|M - K_m$ is a locally flat embedding.

Furthermore, for each $x \in K_m$ there are locally flat m- and q-simplices Δ^m and Δ^q in M and Q respectively, $x \in \Delta^m$, such that

(v) g sends Δ^m piecewise linearly into Δ^q, and

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(vi) the PL structures of K_m and K_q are compatible with those of Δ^m and Δ^q respectively.

DEFINITION. A continuous map $f\colon X \to Y$ is *proper* if the inverse of each compact set is a compact set. A homotopy $f\colon X \times I \to Y$ is *proper* if the inverse of each compact set is compact.

The main result of this paper is:

TOPOLOGICAL GENERAL POSITION LEMMA 1. *Let $f\colon M \to Q$ be a proper map of an m-manifold into a q-manifold, $m \leqslant (2/3)q - 1/3$, $m \leqslant q - 3$ and let $\epsilon\colon M \to (0, 1)$ be a given continuous function. Then there is a general position map $g\colon M \to Q$ such that $d(f(x), g(x)) < \epsilon(x)$, for each $x \in M$, and g is properly homotopic to f.*

DEFINITION. Let $g\colon K \to Q$ be a continuous proper map of a locally finite k-complex into a q-manifold. We say that g is in *general position* if there is a locally finite complex K_q, which is embedded as a closed subset of Q such that

(i) g sends K piecewise linearly onto K_q;

(ii) the singular set of g has dimension $\leqslant 2k - q$;

(iii) for each point $x \in K$ there is a neighborhood, $N(g(x))$, of $g(x)$ and a homeomorphism h_x sending $N(g(x))$ onto a q-simplex such that $h_x \circ g$ is a PL map on some neighborhood of the point x.

GENERAL POSITION LEMMA 2. *Let $f\colon K \to Q$ be a continuous map of a finite k-complex into a q-manifold, $k \leqslant q - 3$, and let $\epsilon > 0$ be given. Then there is a general position map $g\colon K \to Q$ such that $d(f(x), g(x)) < \epsilon$, for each $x \in K$ and g is homotopic to f.*

Furthermore if K' is a subcomplex of K and $f|K'$ is already a general position map then we construct g such that $g|K' = f|K'$.

REMARK. This General Position Lemma 2 is basically a corollary of the codim $\geqslant 3$ Taming Lemma 2.2 and as such it is part of the folklore. (We provide a proof herein because we need and use this lemma in another paper.)

GENERAL POSITION LEMMA 3. *Let M^m and V^v be topological m-and v-manifolds embedded as locally flat closed subsets of a topological q-manifold Q. Let M be compact, m, $v \leqslant q - 3$ and $\varnothing = \partial M = \partial V = \partial Q$. Given an $\epsilon > 0$ there is an ϵ-push P on M in Q and a finite $(m + v - q)$-complex K such that $P(M) \cap V = K$ and each simplex of K is a locally flat subset of M, V and Q.*

Hollingsworth and Sher **[Hl-Sh]** have shown that the triangulation techniques of Kirby and Siebenmann has the following lemma as a consequence:

LEMMA 1.1. *Let K be a finite complex embedded in the interior of a manifold M. Let each simplex of K be a locally flat subset of M,* $\dim K \leqslant 3$, $\dim M \geqslant 5$

and $3 + \dim K \leqslant \dim M$. *Then there is a PL manifold N which is a locally flat subset of M,* $\dim N = \dim M, K \subset \operatorname{Int} N$ *and N is a PL regular neighborhood of K.*

Combining General Position Lemma 3 with Lemma 1.1 and PL block transversality we will establish:

THEOREM 1.2. *Let M and V be topological m-and v-manifolds embedded as locally flat closed subsets of a topological q-manifold Q. Let M be compact, let $w = m + v - q$; suppose*

$$q - 3 \geqslant v \geqslant 5, \quad q - 3 \geqslant m \geqslant 5, \quad w \leqslant 3 \quad \textit{and} \quad \emptyset = \partial M = \partial V = \partial Q.$$

Given an $\epsilon > 0$ there is an ϵ-push P on M in Q such that $W = P(M) \cap V$ is a PL w-manifold, $\partial W = \emptyset$. In addition there are PL block bundles $\eta^{q-v}|W$ and $\xi^{q-m}|W$ and an embedding of their Whitney sum h: $\eta \oplus \xi|W \hookrightarrow Q$ such that

$$h|W = 1, \quad h(\eta^{q-v}|W) \subset P(M) \quad \textit{and} \quad h(\xi^{q-m}|W) \subset V.$$

REMARK. Kirby and Siebenmann have a topological transversality theorem which is valid when the submanifolds do not have dimension 4, but their theorem requires that one of the submanifolds has a normal microbundle neighborhood in Q.

OUTLINE OF PROOF. Here we may apply Lemma 1.1 to the complex $K \subset P(M)$, V and Q of General Position Lemma 3 obtaining PL regular neighborhoods $N^m \subset P(M), N^v \subset V$ and $N^q \subset Q$ of K. Let N^v and N^m be sufficiently small so that they are contained in N^q. Now $N^m \cup_K N^v$ is a *polyhedron* nicely embedded in N^q. Therefore the Taming Lemma 2.2 provides a small push P_1 on N^q such that $P_1(N^m \cup_K N^v)$ is a subcomplex of N^q. Theorem 1.2 follows immediately from this and Rourke and Sanderson's PL block transversality theory [R-S] applied to $P_1(N^m)$ and $P_1(N^v)$ in N^q.

We shall present our proofs in the order of increasing difficulty. General Position Lemma 2 is basically a corollary of the codim $\geqslant 3$ Taming Lemma 2.2. The proof of General Position Lemma 3 is moderately easy. But the proof of Topological General Position Lemma 1 is sufficiently subtle that the author does not know how to establish this lemma outside the metastable (2/3rds) range. In §6 we present two generalizations of Topological General Position Lemma 1.

2. **Background.** Here we shall present some definitions and theorems which will be used in the proof of the General Position Lemmas.

Note. "PL" is a shorthand for piecewise linear.

DEFINITION. A *standard representation* of a neighborhood of a compact subset X, of an n-manifold, M^n, is a collection of compact subsets $\{N_j, I_j, M_j', M_j'', j = 1, 2, \ldots, r\}$ of M such that (see figure):

(1) $N_1 = I_1 = M_1'', M_1' = \emptyset$ and $N_r \supset X$;

(2) I_j is a locally flat n-cell in M;

(3) $X \subset \bigcup_j \operatorname{Int} I_j$;

(4) $M_j' \cup M_j'' = I_j$ and there is a homeomorphism i_j of I_j onto an n-simplex Δ^n such that $i_j(M_j')$ and $i_j(M_j'')$ are combinatorial submanifolds-with-boundary of some rectilinear subdivision of Δ^n;

(5) $N_j \subset N_{j-1} \cup I_j, j > i$;

(6) $\operatorname{Int} N_j \supset X - \bigcup_{m>j} \operatorname{Int} I_m$;

(7) $M_j'' \supset I_j - N_{j-1}$ and $M_j' \subset \operatorname{Int} N_{j-1}, j > 1$;

(8) $M_j'' \cap X \subset \bigcup_{m \geqslant j} \operatorname{Int} I_m$;

(9) $d(M_j'', N_j - I_j) > 0, j > 1$.

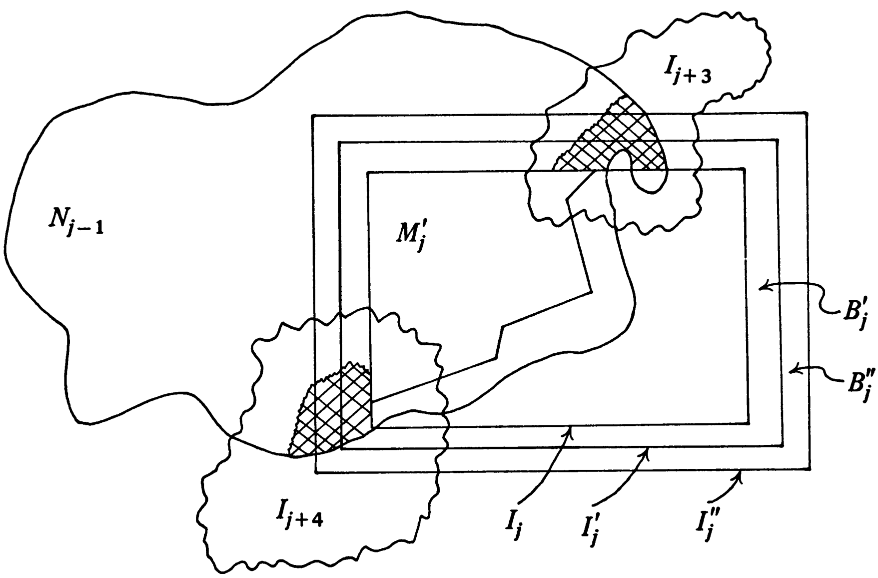
Furthermore, in this paper, we shall let each I_j, $2 \leqslant j \leqslant r$, have three buffer zones or bands B_j, B_j', B_j'' where I_j, I_j', I_j'', B_j' and B_j'' fit together as

$$I_j = [-1, 1]^m, \quad I_j' = [-2, 2]^m, \quad I_j'' = [-3, 3]^m,$$

$$B_j' = I_j' - I_j, \quad B_j'' = I_j'' - I_j' \text{ and } B_j = [-(3/2), 3/2]^m - I_j.$$

Additionally when I_j meets ∂M we want $I_j \subset I_j' \subset I_j'' \supset \partial M \cap I_j''$ to be like (homeomorphic with)

$$[0, 1] \times [-1, 1]^{m-1} \subset [0, 2] \times [-2, 2]^{m-1}$$
$$\subset [0, 3] \times [-3, 3]^{m-1} \supset \{0\} \times [-3, 3]^{m-1}.$$



Note. The shaded area is $N_{j-1} - N_j$

THEOREM 2.1. *For each compact subset X of a manifold there is a standard representation of some neighborhood of X. Furthermore, if $\{I_j, j = 1, 2, \ldots, r\}$ is a particular set of locally-flat n-cells in M such that $X \subset \bigcup_{j=1}^r \operatorname{Int} I_j$, then some neighborhood of X has a standard representation whose collection of I_j's is the given set.*

PROOF. Suppose that X is a compact subset of an n-manifold with boundary M. The hypothesis of Theorem 2.1 or the compactness of X will yield a finite set of locally flat n-cells $\{I_j, j = 1, 2, \ldots, r\}$ in M whose interiors cover X.

Thus (2) and (3) are satisfied.

We begin by setting $N_1 = I_1 = M_1''$, and $M_1' = \emptyset$; thus most of (1) is satisfied.

We will construct the N_j's, M_j''s and M_j'''s by induction. We assume that N_{j-1} is known and satisfies (6). Therefore

$$X \cap \partial N_{j-1} \subset X - \operatorname{Int} N_{j-1} \subset \bigcup_{m \geqslant j} \operatorname{Int} I_m,$$

and

$$N_{j-1} \cup \left(\bigcup_{m \geqslant j} \operatorname{Int} I_m\right) \supset X \supset I_j \cap X.$$

Thus $I_j \cap X$ is contained in the union of two sets, one of which $(\bigcup_{m \geqslant j} \operatorname{Int} I_m) \cap X$ is open in X and contains the boundary in X of the second $(I_j \cap N_{j-1}) \cap X$. Therefore we may find a closed subset A of I_j such that

$$\operatorname{Cl}(I_j - N_{j-1}) \subset \operatorname{Int} A \quad \text{and} \quad A \cap X \subset \bigcup_{m \geqslant j} \operatorname{Int} I_m.$$

Let $B = \operatorname{Cl}(I_j - A)$. Thus $B \subset \operatorname{Int} N_{j-1}$.

One may now use some basic theory of regular neighborhoods (e.g. see Theorem 2.11 of [Hd]) in order to "expand" A and B into "combinatorial submanifolds" M_j'' and M_j' respectively of I_j such that (4), (7) and (8) are satisfied.

Condition (8) implies that there is an open set O_j such that

$$X \cap M_j'' \subset O_j \subset \operatorname{Cl} O_j \subset \bigcup_{m \geqslant j} \operatorname{Int} I_m. \tag{10}$$

We are now ready to define N_j, namely

$$N_j = \operatorname{Cl}(N_{j-1} - O_j) \cup I_j. \tag{11}$$

A brief checking of (10) and (11) will show that (5), (6) and (7) are satisfied. We have shown that all conditions of a standard representation are satisfied. Therefore Theorem 2.1 is established.

ZEEMAN'S UNKNOTTING BALL THEOREM [Z]. *Let $B^m \subset B^q$ be PL m-and-q-balls, $q \geqslant m + 3$ and $\partial B^m = B^m \cap \partial B^q$. Then there is an onto PL homeomorphism*

$$h\colon (I^q = I^m \times I^{q-m}, I^m) \twoheadrightarrow (B^q, B^m).$$

DEFINITION. A map $f\colon K \to Q$ of a polyhedron K into a topological manifold Q is a *nice map* if there is a triangulation of $f(K)$ such that: $f\colon K \to f(K)$

is piecewise linear and for each simplex $\sigma \in K$, $f(\sigma)$ is a locally flat subset of Q.

We shall be using the next lemma in order to "straighten" nice maps.

TAMING LEMMA 2.2 (Bryant, Seebeck, Černavskiĭ, Homma and Miller). *Let K be a k-complex in the interior of a combinatorial q-manifold Q, $k \leqslant q - 3$. Let the interior of each simplex of K be locally flat in Q and let $\epsilon > 0$. Then there is an ambient ϵ-isotopy*

$$\{H_t\colon Q \twoheadrightarrow Q,\ t \in [0, 1] \text{ and } H_0 = 1\}$$

such that

(i) *$H_1 | K$ is piecewise linear, and*

(ii) *$H_t(x) = (x)$, $d(x, K) > \epsilon$ and $t \in [0, 1]$.*

Furthermore, if L is a subcomplex of both K and Q, then we may insist that $H_1 | L = 1$.

REMARK. This Taming Lemma 2.2 is a corollary of the theorems of **[Br-Sb]**, **[Ml]**, **[Cr-2]** and **[Br]**. Of course **[Br-Sb]** uses the ideas of **[Hm]**. The case $k \leqslant (2/3)q - 1$ of Taming Lemma 2.2 is established in **[Cr-1]**.

DEFINITIONS. An *ambient isotopy* of a space Q is a continuous map $H\colon Q \times I \to Q$, such that if we set $h_t(x) = H(x, t)$, then h_t is a homeomorphism of Q onto itself, for each $t \in I$.

We say that h_a is "ambient isotopic" to h_b, for $a, b \in I$.

An *ambient ϵ-isotopy* of Q is an ambient isotopy which satisfies the additional condition:

$$d(h_t(x), x) \leqslant \epsilon, \quad \text{for each } x \in Q,\ t \in I.$$

A *push* is an ambient isotopy of a space Q such that for some compact proper subset A of Q,

$$h_t(x) = x, \quad \text{for all } x \in Q - A \text{ and } t \in I,$$

and $h_0 = 1$. An *ϵ-push* P of A is an ambient ϵ-isotopy of Q such that

$$h_t(x) = x, \quad \text{when } d(x, A) \geqslant \epsilon,\ t \in I$$

and $h_0 = 1$. Also $P(x) = h_1(x)$.

3. **Proof of General Position Lemma 2.** Let f, K and Q satisfy the hypothesis of this lemma. Let $\{I_i, M_i', M_i'', N_i, i = 1, 2, \ldots, r\}$ be a standard representation for Q and let each I_i be contained in a Euclidean neighborhood E_i^q. Let $K_1, \ldots, K_r$ be subcomplexes of K such that

$$\bigcup_{i=1}^{r} K_i = K \quad \text{and} \quad f(K_i) \subset \text{Interior } I_i, \qquad i = 1, 2, \ldots, r.$$

We shall proceed inductively building general position maps g_i: $\bigcup_{j=1}^{i} K_j \to Q$, such that $g_i(K_j) \subset E_j$, $j = 1, 2, \ldots, i$.

First we observe that since $g|K'$ is already a general position map that for each simplex Δ of K' (possibly after subdivision of K') that g sends Δ piecewise linearly into some "patch", interior I^q, in Q. As a consequence of Zeeman's Unknotting Ball Theorem (in §2) we see that $f(\operatorname{Int} \Delta)$ is a locally flat subset of Q. Thus we shall be able to use the Taming Lemma (in §2) on $f|K_i \cap K'$: $K_i \cap K' \to E_i^q$.

Step 1. As just noted we may use Taming Lemma 2.2 in order to obtain a push P_1 on $f(K_1 \cap K')$ such that

$$P_1 \circ f|K_1 \cap K'\colon K_1 \cap K' \to I_1^q$$

is piecewise linear. Let g_1: $K_1 \to I_1^q$ be a general position piecewise linear map which extends $P_1 \circ f|K_1 \cap K'$ and approximates f.

Induction Assumption. Assume that we have constructed g_{i-1}: $K' \cup \bigcup_{j=1}^{i-1} K_j \to Q$ as a general position map such that

$$g_{i-1}|K' = f|K' \quad \text{and} \quad g_{i-1}(K_j) \subset E_j^q, \qquad j = 1, 2, \ldots, i-1.$$

Step i. Let $K_i' = K_i \cap (K' \cup \bigcup_{j=1}^{i-1} K_j)$. As before we may use the Taming Lemma 2.2 in order to obtain a push P_i on $g_{i-1}(K_i')$ in E_i^q so that $P_i \circ g_{i-1}|$: $K_i' \to E_i^q$ is piecewise linear. Let g_i': $K_i \to E_i^q$ be a general position piecewise linear map which extends $P_i \circ g_{i-1}|K_i'$ and approximates f.

Let g_i: $K' \cup \bigcup_{j=1}^{i} K_j \to Q$ be defined by

$$g_i(x) = \begin{cases} P_i^{-1} \circ g'(x), & x \in K, \\ g_{i-1}(x), & x \in K' \cup \bigcup_{j=1}^{i-1} K_j. \end{cases}$$

Clearly g_i is a well-defined general position map of the type needed for our induction. So the induction works,

$$g_r\colon K = K' \cup \bigcup_{i=1}^{r} K_i \to Q, \qquad g_r|K' = f|K'$$

and g_r is a general position map.

Since the euclidean patches E_i^q are convex g_r will be homotopic to f. This completes the proof of General Position Lemma 2.

REMARK. In a manner analogous to the one we shall use at the end of the proof of the Topological General Position Lemma 1, it is easily shown that General Position Lemma 2,is also valid when f: $K \to Q$ is a proper map and K is a locally finite complex.

4. **Proof of General Position Lemma 3.** Let M, V and Q be as in the statement of this lemma. Choose a standard representation $\{N_j^v, I_j^v, V_j', V_j'', j = 1, 2,$

$\ldots, s\}$ of a neighborhood of the compact subset $M \cap V$ of the manifold V such that each I_j^v has a euclidean neighborhood $E_j^q = E^v \times E^{q-v}$ where $1\colon I_j^v \hookrightarrow E^v$ and diam $E_j^q < \epsilon$ (courtesy of Theorem 2.1). That we can find such a collection $\{I_j^q\}$ follows from the local flatness of V in Q.

We shall perform an induction on these N_j^v's of this standard representation as follows:

INDUCTIVE STATEMENT A_n. There is a set of n pushes $\{P_1, \ldots, P_n\}$ where each P_j is an (ϵ/s)-push on V_j'' in E_j^q such that Int N_n^v and $P_n \circ P_{n-1} \circ \cdots \circ P_1(M)$ are in general position and

$$V \cap P_n \circ P_{n-1} \circ \ldots \circ P_1(M) \subset N_s^v.$$

Thus statement A_s will establish the lemma.

Preliminaries for proof that statement A_{n-1}, implies statement A_n, $n = 2, \ldots, s$. Let

$$K_{n-1} = P_{n-1} \circ \ldots \circ P_1(M) \cap \mathrm{Int}(N_{n-1}^v).$$

Statement A_{n-1} implies that K_{n-1} is a locally finite complex which has a nice proper embedding in Int N_{n-1}^v.

Now K_{n-1} has a subcomplex K_{n-1}^* which "approximates" $K_{n-1} \cap V_n'$ in the sense that $K_{n-1} \cap V_n' \subset K_{n-1}^* \subset K_{n-1} \cap E_n^v$. Therefore Taming Lemma 2.2 provides a push on K_{n-1}^* in E_n^v which sends K_{n-1}^* onto a polyhedron in E_n^v. Therefore without loss of generality we may assume that K_{n-1}^* is a subcomplex of E_n^v and hence $K_{n-1} \cap V_n'$ is a subcomplex of subdivisions of both K_{n-1} and V_n'.

This completes the *preliminaries* for showing that statement A_{n-1} implies statement A_n.

Now for each integer $n = 1, 2, \ldots, s$, there is a standard representation

$$\{N_{n,i}^m, I_{n,i}^m, M_{n,i}', M_{n,i}'', i = 1, 2, \ldots, k_n\}$$

of $P_{n-1} \circ \cdots \circ P_1(M) \cap V_n''$ in M. We shall perform an inner induction on these $N_{n,i}^m$'s as follows:

INDUCTIVE STATEMENT $B_{n,i}$. There is a collection

$$\{P_{n,j}' \text{ is an } (\epsilon/sk_n)\text{-push on the image of } M_j'' \text{ in } E_n^q, j = 1, 2, \ldots, k_n\}$$

and, if we set

$$g_{n,i} = P_{n,i}' \circ \ldots \circ P_{n,1}' \circ P_{n-1} \circ \ldots \circ P_1,$$

then $g_{n,i}(\mathrm{Int}\, N_{n,i}^m)$ and E_n^v are in general position in E_n^q. In addition

$$I_n^v \cap g_{n,i}(M) \subset (K_{n-1}^* \cap V_n') \cup g_{n,i}(N_{n,k_n}^m)$$

and $g_{n,i}$ agrees with P_{n-1} on $N_n^v - V_n''$.

Thus for each n the statement B_{n,k_n} will establish the statement A_n so we now show that statement $B_{n,i-1}$ implies statement $B_{n,i}$. For this we work entirely in E_n^q where $V \cap E_n^q = E_n^v$, a v-dimensional vector subspace.

Statement $B_{n,i-1}$ implies that there is a finite subcomplex $L_{n,i-1}^*$ of $E_n^v \cap g_{n,i-1}(\operatorname{Int} N_{n,i-1}^m)$ which "approximates" $E_n^v \cap g_{n,i-1}(M'_{n,i})$ in the sense that

$$E_n^v \cap g_{n,i-1}(M'_{n,i}) \subset L_{n,i-1}^* \subset E_n^v \cap g_{n,i-1}(\operatorname{Int} N_{n,i-1}^m)$$

and $L_{n,i-1}^*$ is nicely embedded in V, Q and the image of M. Therefore the Taming Lemma 2.2 provides a small push on $g_{n,i-1}^{-1}(L_{n,i-1}^*) \cap M'_{n,i}$ in M which pushes a small neighborhood of $g_{n,i-1}^{-1}(L_{n,i-1}^*) \cap M'_{n,i}$ onto a subpolyhedron of a euclidean neighborhood of $I_{n,i}^m$. Therefore without loss of generality we may assume that the set defined as

$$L_{n,i-1} = L_{n,i-1}^* \cap g_{n,i-1}(M'_{n,i}) = E_n^v \cap g_{n,i-1}(M'_{n,i}) \tag{4.1}$$

is a subcomplex of both $L_{n,i-1}^*$ and $g_{n,i-1}(M'_{n,i})$.

Therefore $L_{n,i-1}$ is nicely embedded in E_n^v and, by the Taming Lemma 2.2 again, we may assume, without loss of generality, that $L_{n,i-1}$ is also a subcomplex of E_n^v.

The clever idea of this proof is that we have not arranged things (with the aid of (4.1)) so that the set

$$R_{n,i-1} \overset{\text{def}}{=} E_n^v \cup g_{n,i-1}(M'_{n,i})$$

is a *complex* nicely embedded in E_n^q.

The Taming Lemma 2.2 provides an $(\epsilon/2si_n)$-push $P''_{n,i}$ of $g_{n,i-1}(M'_{n,i})$ in E_n^q such that

(a) $P''_{n,i}$ moves $R_{n,i-1}$ onto a subcomplex of E_n^q;
(b) $P''_{n,i}$ moves *no* point of E_n^v; and
(c) $P''_{n,i} \circ g_{n,i-1}|\colon M'_{n,i} \to E_n^q$ is PL.

Now the Taming Lemma 2.2 provides an $(\epsilon/2si_n)$-push $P'''_{n,i}$ of $P''_{n,i}g_{n,i-1}(M''_{n,i})$ in E_n^q such that

$$P'''_{n,i}P''_{n,i}g_{n,i-1}|\colon I_{n,i}^m \to E_n^q$$

is PL and in PL-general-position with respect to E_n^v and such that

$$P'''_{n,i}|(P''_{n,i}g_{n,i-1}(N_{n,i}^m - M''_{n,i})) \cap E_n^v = 1.$$

Now $P'_{n,i} = P'''_{n,i} \circ P''_{n,i}$ is the push demanded by statement $B_{n,i}$ since both $P''_{n,i}$ and $P'''_{n,i}$ preserve the intersection of E_n^v with the image of $M'_{n,i}$ and $P'''_{n,i}$ places the image of $M''_{n,i}$ in general position with respect to E_n^v.

This establishes statement $B_{n,i}$, $i = 1, 2, \ldots, i_n$. We set $P_n = P'_{n,k_n} \circ \cdots \circ P'_{n,2} \circ P'_{n,1}$ and then statements $A_n, n = 1, 2, \ldots, s$, are established.

Finally statement A_s establishes General Position Lemma 3.

5. **Proof of Topological General Position Lemma 1.** In this section we shall first establish the Topological General Position Lemma 1 for the case when M is compact and $\epsilon(x)$ is a constant function, i.e. $\epsilon(x) = \epsilon > 0$. Following this we shall indicate how the noncompact case follows from the compact case.

The epsilontics will be omitted from the proof. The reader may easily fill them in.

PROOF OF TOPOLOGICAL GENERAL POSITION LEMMA 1. *Case* (i) M is compact.

Let f, M, and Q satisfy the hypotheses of the lemma and let M be compact. Let $\{I_i, M_i', M_i'', N_i, i = 1, \ldots, r\}$ be a standard representation of M such that each $f(I_i'')$ is contained in some Euclidean open set E_i^q of Q (see Theorem 2.1).

The proof is a double induction, namely;

BIG INDUCTION. We shall construct approximations g_i: $M \to Q$ of f, $i = 1, 2, \ldots, r$, such that $g_i|\mathrm{Int}\, N_i$ is a general position map without triple points.

LITTLE INDUCTION. We shall construct approximations g_{ij}: $M \to Q$ of g_{i-1}, $j = 1, 2, \ldots, i-1$, and small pushes P_{ij} of $(Q, g_{i,j-1}(I_j))$, $j = 1, \ldots, i-1$ such that

(i) $g_{ij}|\mathrm{Int}\, N_j \cup I_i - B_i'$ is a general position map; and

(ii) $g_{ij}|I_i = g_{i,1}|I_i$ $(= g_i I_i)$ is a piecewise linear general position map of I_i into $E_i^q \subset Q$;

(iii) $g_{ij}|N_i - I_i = P_{ij}g_{i,j-1}|N_i - I_i$;

(iv) $g_{ij}|\mathrm{Int}\, N_i$ has no triple points;

(v) $g_{i,j}(x) = g_{i,j-1}(x)$, $x \in M - \bigcup_{k \leq i} I_k$ and $j > 1$.

All that we have to do in order to prove General Position Lemma 1 is to indicate how to do the steps of this double induction. Here goes: (note: The subscripts i and j will refer to the Big Induction and the Little Induction, respectively).

Step $i = 1$. Approximate $f|I_1$: $I_1 \to E_1^q \subset Q$ by a piecewise linear general position map. Extend this map to all of M obtaining g_1: $M \to Q$.

Step $i = 2$. *Beginning.* (i) Let S_1 be the singularities of $g_1|N_1$. Then there is a subpolyhedron K of I_1 such that $S_1 \cap I_2' \subset K \subset S_1 \cap I_2''$. One may use Taming Lemma 2.2 in order to find a small push P of (I_2'', K) such that $P(K)$ is a subpolyhedron of I_2''. Therefore without loss of generality we may assume that $S_1 \cap I_2'$ is a subpolyhedron of I_2' and that the triangulations of I_1 and I_2' are compatible on $S_1 \cap I_2'$.

(ii) We can construct a small push P of $(Q, g_1(M_2'))$ such that $Pg_1|M_2'$: $M_2' \to E_2^q \subset Q$ is a piecewise linear map. This follows from Taming Lemma 2.2, Steps $i = 1$ and (i) above. Thus, without loss of generality we may assume that $g_1|M_2'$: $M_2' \to E_2^q \subset Q$ is a piecewise linear map.

(iii) Extend $g_1|M_2'$ to a piecewise linear general position map g_2: $I_2 \to E_2^q$, such that

$$g_2(I_2 - M_2') \cap g_1((S_1 - I_2) \cap N_2) = \emptyset \tag{5.1}$$

using the dimension restriction $m \leqslant (2/3)q - (1/3)$ and Lemma 5 of [D], namely.

LEMMA 5.1. *Let f be a map of an r-complex K into a combinatorial n-manifold M. Let X be the finite union of tame s-complexes in M, $r + s < n$. If $\epsilon > 0$ is given, then there is a piecewise linear map g: $K \to M$ such that $g(K) \cap X = \emptyset$ and $d(g, f) < \epsilon$.*

Furthermore, if L is a subcomplex of K such that $f|L$ is already piecewise linear and $f(L) \cap X = \emptyset$, then $g|L = f|L$.

We shall finish Step $i = 2$ after we complete:

Step $i = 2$ and $j = 1$. Let K_2' be a subpolyhedron of I_1 such that

$$\text{Closure}[I_1 - I_2'] \cap f^{-1}(E_2^q) \subset K_2' \subset [I_1 - I_2 - B_2] \cap f^{-1}(E_2^q).$$

Let $K_2 = K_2' \cup M_2'$. Thus K_2 is a polyhedron with K_2' and M_2' as disjoint subpolyhedra.

A little checking shows that $g_1|K_2$ is a nice map. Therefore, we may use Taming Lemma 2.2 in order to obtain a small push $P_{2,1}$ of $(Q, g_1(K_2'))$ such that:

(i) $P_{2,1}g_1|K_2$: $K_2 \to E_2^q$ is piecewise linear,

(ii) $P_{2,1}g_1|K_2$ is in general position with respect to $g_2|I_2$, and

(iii) $P_{2,1}|g_1(M_2') = 1$.

Thus we may easily construct $g_{2,1}$: $M \to Q$ as an extension of $P_{2,1} \circ g_1|K_2$ and $g_2|I_2$. This $g_{2,1}$ together with $P_{2,1}$ will satisfy the Little Induction hypothesis for $i = 2$ and $j = 1$.

Step $i = 2$. Conclusion. All that remains is to "correct" the intersection of $g_{2,1}(B_2')$ with $g_{2,1}(M_2'')$.

One may use an argument which is similar to the beginning of Step $i = 2$ in order to show that we may assume without loss of generality that:

(i) The singularities of $g_{2,1}|K_2 \cup I_2$ intersect B_2'' (as well as I_2) as a subpolyhedron of I_2''.

(ii) Let $K_2'' = (B_2'' \cap \text{Int}\, N_1) \cup I_2$. Then $g_{2,1}|K_2''$: $K_2'' \to E_2^q \subset Q$ is a piecewise linear map.

As a consequence of equation (5.1) there exists a closed neighborhood U_2 of $[g_{2,1}^{-1}g_{2,1}(M_2'')] \cap B_2' \cap N_2$ such that $U_2 \cap (S_1 \cup M_2'' \cup (M - N_1)) = \emptyset$ and $K_2'' \cup U_2$ is a combinatorial submanifold of I_2''. (Remember that S_1 is the singularities of $g_1|N_1$.)

REMARK. We only need the dimension restriction $m \leqslant (2/3)q - (1/3)$ in order to obtain this set U_2 via equation (5.1).

Let K_2''' be a subpolyhedron of I_2'' such that

$$(U_2 \cup M_2' \cup B_2'') \cap N_2 \subset K_2''' \subset (U_2 \cup M_2' \cup B_2'') \cap N_1.$$

Since $g_{2,1}|K_2''' = P_{2,1}g_1|K_2'''$, we see that $g_{2,1}$ is a nice map of K_2''' into E_2^q. Therefore there is a push P_2 of $(E_2^q, g_{2,1}(U_2))$ such that

$P_2 g_{2,1}|U_2\colon\ U_2 \to E_2^q \subset Q$ is piecewise linear,

$P_2|g_{2,1}(K_2''' - U_2) \cup (N_1 - I_2'') = 1,$

$P_2 g_{2,1}|U_2$ is in general position with respect to $g_{2,1}M_2''$.

A little checking will show that the desired map is

$$g_2(x) = \begin{cases} g_{2,1}(x), & x \in (M - N_1) \cup I_2, \\ P_2 g_{2,1}(x), & x \in N_1. \end{cases}$$

This completes Step $i = 2$.

Step $i = 3$. *Beginning.* This is essentially the same as the beginning of Step $i = 2$. Thus we obtain a piecewise linear map $g_3\colon I_3 \to E_3^q$ such that

(i) $g_3(I_3 - M_3') \cap g_2(S_2 - M_3') \cap N_3 = \varnothing$, and without loss of generality that

(ii) $S_2 \cap I_3'$ is a subpolyhedron of I_3',

(iii) $g_3|M_3' = g_2|M_3'$,

where S_2 is the set of singularities of $g_2|N_2$.

Step $i = 3, j = 1$. This is essentially the same as Step $i = 2$ and $j = 1$ for $I_1 \cap N_2$ and I_3 instead of I_1 and I_2, respectively. Some minor problems appear at $I_1 \cap \partial N_2$. These problems should be ignored since $I_1 \cap \partial N_2 \subset \bigcup_{k>1} I_k$, and hence these bad points will automatically disappear later.

Step $i = 3$ *and* $j = 2$. Let $K_{3,2}$ be a subpolyhedron of $I_3 \cup (M_2' - (I_3 \cup B_3))$ such that $I_3 \cup M_2' - B_3' \subset K_{3,2}$. We observe that $g_{3,1}|K_{3,2}$ is a nice map of $K_{3,2}$ into E_3^q. Hence, as before without loss of generality, we temporarily assume that $g_{3,1}|K_{3,2}$ is piecewise linear.

Let $K_{3,2}'$ be a polyhedron which "approximates" $(I_2 - I_3) \cup M_3'$ namely $K_{3,2}'$ is a subpolyhedron of $[I_2 - (I_3 \cup B_3)] \cup M_3'$ which contains $(I_2 - I_3') \cup M_3'$.

We see that $g_{3,1}|K_{3,2}'$ is a nice map. Hence the Taming Lemma 2.2 applies. Therefore there is a push $P_{3,2}$ on $(Q, g_{3,1}(I_2 - M_3'))$ such that

(i) $P_{3,2}g_{3,1}|K_{3,2}'$ is piecewise linear;

(ii) $P_{3,2}|g_{3,1}(M_3') = 1$;

(iii) $P_{3,2}g_{3,1}|K_{3,2}'$ is in general position with respect to $g_3|I_3$.

The reader may now easily finish the construction of $g_{3,2}\colon M \to Q$ as an extension of $P_{3,2} \circ g_{3,1}|K_{3,2}'$ and $g_{3,1}|I_3$; this $g_{3,2}$ together with $P_{3,2}$ above will satisfy the conditions of the Little Induction for $i = 3$ and $j = 2$.

Step $i = 3$. *Conclusion.* This is essentially the same as the conclusion of Step $i = 2$.

This completes the proof of Step $i = 3$ of the Big Induction.

By continuing in this manner the proof of the General Position Lemma will be completed. All the steps in each of the following collections are essentially the same:

(i) Beginning of Step $i, i \geqslant 2$,

(ii) Steps $j = 1$ for $i \geqslant 2$,

(iii) Steps $j \geqslant 2$ for $i \geqslant 3$,

(iv) Conclusion of Step $i, i \geqslant 2$.

Remark on Steps $j \geqslant 2$ and $i \geqslant 4$. We note that

$$K_{ij} \text{ will approximate } I_i \cup (M'_j - I'_i - N^c_{i-1});$$

$$K'_{ij} \text{ will approximate } M'_i \cup (I_j - I'_i - N^c_{i-1}).$$

There are some difficulties at the boundary of N_{i-1}. These should be ignored since $\partial N_{i-1} \subset \bigcup_{k \geqslant i} I_k$ and hence will automatically be corrected later.

The final map g_r: $M \to Q$ is the map demanded by General Position Lemma 1.

Finally we observe that, since every manifold is locally contractible, if g is sufficiently close to f then g is homotopic to f. This completes the proof for the case M is compact.

Case (ii). M is not compact. Here $M = \bigcup_{i=1}^{\infty} X_i$ where each X_i is compact and $X_i \subset \operatorname{Int} X_{i+1}, i = 1, 2, \ldots$. Let $\epsilon_i = \min\{\epsilon(x) | x \in X_i\}$.

So we construct g on M by constructing a sequence of g_i's on neighborhoods of the X_i's with the aid of standard representations for neighborhoods of the X_i's in M. These g_i's will be general position maps on some small neighborhoods of the X_i's and it is easily arranged that $d(g_i(x), f(x)) < \epsilon_i$, $x \in X_i - X_{i-1}$.

Furthermore, since f is a proper map for each integer i_0 there is an integer i_1 such that $f(X_{i_0}) \cap f(M - X_{i_1}) = \varnothing$.

Hence we may arrange the g_i's so that $g_{i_1} | X_{i_0} = g_n | X_{i_0}$ for each $n > i_1$. Therefore the g_i's will converge to a continuous function g which satisfies all the conditions of a general position map.

Again we observe that, since every manifold is locally contractible, if we keep g sufficiently close to f, then g will be properly homotopic to f.

This completes the proof of the Topological General Position Lemma 1.

6. Corollaries of Topological General Position Lemma 1. In this section we shall establish two corollaries of (the proof of) the Topological General Position Lemma 1. We need and use these two corollaries in papers on topological embeddings which we are currently writing.

COROLLARY 6.1. *Let f: $(M, \partial M) \to (Q, \partial Q)$ be a continuous proper map of an m-manifold into a q-manifold, $m \leqslant (2/3)q - (1/3)$, $m \leqslant q - 3$ and let ϵ: $M \to (0, 1)$ be a given continuous function. Then there is a general position map g: $M \to Q$ such that $g|$: $\partial M \to \partial Q$ is also a general position map,*

$$d(f(x), g(x)) < \epsilon(x), \quad \textit{for each } x \in M,$$

and g is properly homotopic to f.

PROOF. The Topological General Position Lemma 1 provides a general position map $g_1|$: $\partial M \to \partial Q$ which approximates $f|\partial M$. The collars of ∂M in M and of ∂Q in Q enables one to extend this map g_1 to a general position map $g_1 \times 1$ defined on an open collar of ∂M into a small collar of ∂Q so that $g_1 \times 1$ approximates f on the open collar of ∂M. Let f_1 be a close approximation of f which agrees with $g_1 \times 1$ on the domain of $g_1 \times 1$. A general position map g: $(M, \partial M) \to (Q, \partial Q)$ which approximates f_1 and which agrees with f_1 and $g_1 \times 1$ on some small closed collar of ∂M is obtained by applying the proof of the Topological General Position Lemma 1 to f_1 on M minus the domain of $g_1 \times 1$. Thus Corollary 6.1 is established.

COROLLARY 6.2. *Let f: $M \to Q$ be a proper map of a topological m-manifold into a topological q-manifold, $m \leqslant (2/3)q - 1/3$, and let ϵ: $M \to (0, \infty)$ be a continuous function. Then there is a proper general position map g: $M \to Q$ such that $d(f(x), g(x)) \leqslant \epsilon(x)$.*

Furthermore if $f|\partial M$ is already a locally flat embedding then g may be constructed so that $g|\partial M = f|\partial M$.

Note. Here $f(M)$ *is still* permitted to intersect both the interior of Q and $f(\text{Int } M)$.

The proof of Corollary 6.2 is a slightly complicated variation of the proof of the Topological General Position Lemma 1.

OUTLINE OF THE PROOF OF COROLLARY 6.2. Let f: $M \to Q$ satisfy the hypothesis of this Corollary 6.2. We use here the same standard representation and the same big and little induction statements as used in the proof of the Topological General Position Lemma 1. These inductions here have an analogous set of "Steps" to the previous set of steps.

Step $i = 1$. First push $f(I_1 \cap \partial M)$ onto a subpolyhedron of E_1^q by some small push P_1^*; this is possible because $f|\partial M$ is a locally flat embedding, locally flat embeddings are locally tame and because of the Taming Lemma 2.2. Now approximate $P_1^* f|$: $I_1 \to E_1^q$ by a PL general position map f^*: $I_1 \to E_1^q$ which extends $P_1^* f|I_1 \cap \partial M$. Extend $(P_1^*)^{-1} f^*$ to all of M obtaining g_1: $M \to Q$. Note that on $I_1 \cap \partial M$, g_1 agrees with $(P_1^*)^{-1} P_1^* f = f$.

Step $i = 2$. Beginning. Parts (i) and (ii) are the same as in the proof of the Topological General Position Lemma 1.

(iii) First use the Taming Lemma 2.2 in order to push $g_1(\partial M \cap M_2'')$ onto a subpolyhedron of E_2^q by a push P_2^* which moves no point of $g_1(\partial M \cap M_2')$ such that

$$[P_2^* g_1(\partial M \cap M_2'')] \cap g_1((S_1 - I_2) \cap N_2) = \varnothing$$

and $P_2^* g_1 | \partial M \cap I_2$ is in PL general position with respect to $g_1 | M_2'$ using the dimension restriction $m \leqslant (2/3)q - (1/3)$ and

LEMMA 6.3. *If the f of Lemma* 5.1 *is a locally tame embedding and $r \leqslant n - 3$, then the g of Lemma* 5.1 *may be constructed so that there is an ϵ-push P^* such that $g = P^* f$ and P^* moves no point of L.*

REMARK. Lemma 6.3 is a corollary of the proof of Lemma 19 of [D] and the Taming Lemma 2.2.

Now extend $P_2^* g_1 | \partial M \times I_2$ and $g_1 | M_2'$ to a PL general position map $g_2 \colon I_2 \to E_2^q$ such that

$$g_2(I_2 - M_2') \cap g_1((S_1 - I_2) \cap N_2) = \varnothing$$

using Lemma 5.1 and the hypothesis $m \leqslant (2/3)q - (1/3)$.

Note. $g_2 | \partial M \times I_2 = P_2^* f | \partial M \times I_2$.

This completes Step $i = 2$. Beginning.

We would like to use Step $i = 2$ and $j = 1$ of the proof of the Topological General Position Lemma here while being careful that

$$P_{2,1} g_1(\partial M \cap (I_1 - I_2)) \cap P_2^* g_1(\partial M \cap M_2'') = \varnothing.$$

Unfortunately, we do not know how to guarantee that $g_{2,1} | \partial M$ will have no self intersection at $N_1 - N_2$. To overcome this problem we shall run through the previous Step $i = 2$ and $j = 1$ twice; the first time we will adjust just the image of $\partial M \cap K_2'$ with respect to the image of I_2, without destroying the fact that ∂M is embedded in Q, and the second time to adjust the image of K_2' while fixing the image of $\partial M \cap K_2'$.

Step $i = 2$ and $j = 1$. Let K_2' be as in Step $i = 2$ and $j = 1$ of the proof of Topological General Position Lemma. Let $(K_2')^* = K_2' \cap \partial M$ and $K_2^* = (K_2')^* \cup M_2'$. As in the previous Step $i = 2$ and $j = 1$, the Taming Lemma 2.2 provides a small push $P_{2,1}^*$ of $(Q, g_1((K_2')^*))$ such that $P_{2,1}^* g_1 | K_2^*$ is PL and is in general position with respect to $g_2 | I_2$ and

$$P_{2,1}^* | P_2^* g_1(M_2') \cup [\partial M \cap (B_1'' \cup B_2 \cup I_2)] = 1.$$

Here $P_{2,1}^*$ is a δ-push where

$$2\delta > d(g_1(K_2')^*, g_1(\partial M - (K_2') - B_1'' - B_2 - I_2)).$$

Therefore, if $x, y \in \partial M$ and $P_{2,1}^* g_1(x) = g_1(y)$ then both x and y must be in $(K_2')^* \cap (\partial M \cap B_1')$.

Now let $(K_2')^{**}$ be a subpolyhedron of K_2' which approximates

$$K_2' - g_1^{-1}[g_1(S_1 \cap (N_1 - N_2 - B_2'))].$$

Using $(K_2')^{**}$ instead of K_2' in the procedure of Step $i = 2$ and $j = 1$ of the Topological General Position Lemma yields a small push $P_{2,1}^{**}$ such that

$$P_{2,1}^{**} P_{2,1}^* P_2^* g_1 | K_2 - [K_2' - (K_2')^{**}] \text{ is PL}$$

and is in general position with respect to $g_2|I_2$ and such that

$$P_{2,1}^{**} | P_{2,1}^* P_2^* g_1 (M_2' \cup (\partial M \cap (I_1 - I_2 - B_2'))) = 1.$$

By using $(K_2')^{**}$ instead of K_2' above we were able to avoid moving the image of $S_1 \cap (N_1 - N_2 - B_2') \cap \partial M$.

Therefore we may easily construct $g_{2,1}: M \to Q$ as an extension of

$$P_{2,1}^{**} P_{2,1}^* P_2^* g_1 | K_2, \quad g_2 | I_2 \quad \text{and} \quad P_{2,1}^* P_2^* g_1 | \partial M.$$

Note. Since $g_{2,1} | \partial M = P_{2,1}^* P_2^* g_1 | \partial M$, it is a locally flat embedding.

Step $i = 2$. Conclusion. This is the same as Step $i = 2$ (conclusion) of the proof of the Topological General Position Lemma 1 with the addition that here P_2 is a δ-push where

$$2\delta > d(g_2(\partial M \cap (N_2 - I_2), g_2(\partial M - N_1)).$$

This choice of δ yields that $g_2 | \partial M$ is a locally flat embedding. A little checking will show that $g_2 | \partial M$ was obtained from $g_1 | \partial M$ by three small pushes.

Step $i = 3$ and $j = 2$. This step begins the same as Step $i = 3$ and $j = 2$ of the proof of the Topological General Position Lemma 1 with the same $K_{3,2}$ and also having $g_{3,1} | K_{3,2}$ being PL. In the proof of Topological General Position Lemma 1, the remainder of Step $i = 3$ and $j = 2$ is just like Step $i = 2$ and $j = 1$. So we complete this step here by retracing our path through Step $i = 2$ and $j = 1$.

As before, the other steps in the inductions are analogous to the ones we just discussed. The result of going through all the steps is a general position map $g_r: M \to Q$ such that $g_r | \partial M$ was obtained from $f | \partial M$ by a series of small pushes. If P is the composition of these pushes then $g_r | \partial M = Pf | \partial M$ and hence $f | \partial M = P^{-1} g_r | \partial M$. Therefore $P^{-1} g_r: M \to Q$ is the desired general position approximation to f. Thus Corollary 6.2 is established.

Added in proof. We mention here two more corollaries of the techniques of this paper. We shall use both of these corollaries as lemmas in another paper that we are currently writing.

COROLLARY 6.3. *Let $f: M \to Q$ be a proper map of a topological m-manifold into a topological q-manifold, $m \leqslant (2/3)q - 1/3$ and let $\epsilon: M \to (0, \infty)$ be a continuous function. Suppose $\partial M = M_1 \cup M_2$, where M_1 and M_2 are two disjoint $(n-1)$-submanifolds of ∂M, $\partial M_1 = \varnothing = \partial M_2$. Suppose $f|M_1$ is a locally flat embedding. Then there is a proper general position map $g: M \to Q$ such that $d(f(x), g(x)) < \epsilon(x)$, for each $x \in M$, and $g|M_1 = f|M_1$.*

REMARK. Corollary 6.3 is an immediate consequence of our proof of Corollary 6.2 (just treat M_2 as if it is part of the interior of M).

COROLLARY 6.4. *Let V be a proper locally flat topological v-submanifold in the interior of a topological q-manifold Q, $q \geqslant v + 3$. Let K be a locally finite simplicial k-complex, $k \leqslant q - 3$, which is properly locally-tamely embedded in Q. Then there is an ambient isotopy*

$$\{H_t: Q \to Q,\ H_0 = 1\}$$

such that $H_1(K) \cap V$ is a $(v + k - q)$-subcomplex K_2 of $H_1(K)$ and K_2 is locally-tamely embedded in V.

REMARK. Corollary 6.4 is a corollary of the proof of General Position Lemma 3. The proof of Corollary 6.4 is actually simpler because of the triangulation of K. Therefore, here the standard representation of M, in the proof of General Position Lemma 3 (when M is compact), is replaced by a set of subcomplexes $\{K_1, \ldots, K_n\}$ of K such that $K = \bigcup_{j=1}^n K_j \subset E_j^q$, $j = 1, 2, \ldots, n$.

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